

# Note on the Sum of Powers of Signless Laplacian Eigenvalues of Graphs

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## Abstract

For a simple graph  $G$  and a real number  $\alpha$  ( $\alpha \neq 0, 1$ ) the graph invariant  $s_\alpha(G)$  is equal to the sum of powers of signless Laplacian eigenvalues of  $G$ . In this note, we present some new bounds on  $s_\alpha(G)$ . As a result of these bounds, we also give some results on incidence energy.

## 1 Introduction

Let  $G$  be a simple graph with  $n$  vertices and  $m$  edges. Let  $V(G) = \{v_1, v_2, \dots, v_n\}$  be the set of vertices of  $G$ . For  $v_i \in V(G)$ , the degree of the vertex  $v_i$ , denoted by  $d_i$ , is equal to the number of vertices adjacent to  $v_i$ . Throughout this paper, the maximum, the second maximum and the minimum vertex degrees of  $G$  will be denoted by  $\Delta_1$ ,  $\Delta_2$  and  $\delta$ , respectively.

Let  $A(G)$  be the  $(0, 1)$ -adjacency matrix of a graph  $G$ . The eigenvalues of  $G$  are the eigenvalues of  $A(G)$  [6] and denoted by  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ . Then the energy of a graph  $G$  is defined by [17]

$$E = E(G) = \sum_{i=1}^n |\lambda_i|.$$

There is an extensive literature on this topic. For more details see [18, 28] and the references cited therein.

The concept of graph energy was extended to energy of any matrix in the following manner [36]. Recall that the singular values of any (real) matrix  $M$  are equal to the square roots of the eigenvalues of  $MM^T$ , where  $M^T$  is the transpose of  $M$ . Then

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the energy of the matrix  $M$  is defined as the sum of its singular values. Clearly,  $E(A(G)) = E(G)$ .

Let  $D(G)$  be the diagonal matrix of vertex degrees of  $G$ . Then the Laplacian matrix of  $G$  is  $L(G) = D(G) - A(G)$  and the signless Laplacian matrix of  $G$  is  $Q(G) = D(G) + A(G)$ . As well known in spectral graph theory, both  $L(G)$  and  $Q(G)$  are real symmetric and positive semidefinite matrices, so their eigenvalues are non-negative real numbers. Let  $\mu_1 \geq \mu_2 \geq \dots \geq \mu_n = 0$  be the eigenvalues of  $L(G)$  and let  $q_1 \geq q_2 \geq \dots \geq q_n$  be the eigenvalues of  $Q(G)$ . These eigenvalues are called Laplacian and signless Laplacian eigenvalues of  $G$ , respectively. For details on Laplacian and signless Laplacian eigenvalues, see [7–10, 33, 34].

The incidence matrix  $I(G)$  of a graph  $G$  with the vertex set  $V(G) = \{v_1, v_2, \dots, v_n\}$  and edge set  $E(G) = \{e_1, e_2, \dots, e_m\}$  is the matrix whose  $(i, j)$ -entry is 1 if the vertex  $v_i$  is incident with the edge  $e_j$ , and is 0 otherwise. In [24], Jooyandeh et al. motivated the idea in [36] and defined the incidence energy of  $G$ , denoted by  $IE(G)$ , as the sum of singular values of  $I(G)$ . Since  $Q(G) = I(G)I(G)^T$ , it was later proved that [20]

$$IE = IE(G) = \sum_{i=1}^n \sqrt{q_i}$$

For the basic properties of  $IE$  involving also its lower and upper bounds, see [3, 4, 13, 20, 21, 24, 32, 38, 42, 43].

In [30] Liu and Lu introduced a new graph invariant based on Laplacian eigenvalues

$$LEL = LEL(G) = \sum_{i=1}^{n-1} \sqrt{\mu_i}$$

and called it Laplacian energy like invariant. At first it was considered that [30]  $LEL$  shares similar properties with Laplacian energy [22]. Then it was shown that it is much more similar to the ordinary graph energy [23]. For survey and details on  $LEL$ , see [29].

For a graph  $G$  with  $n$  vertices and a real number  $\alpha$ , to avoid trivialities it may be required that  $\alpha \neq 0, 1$ , the sum of the  $\alpha$ th powers of the non-zero Laplacian eigenvalues is defined as [41]

$$\sigma_\alpha = \sigma_\alpha(G) = \sum_{i=1}^{n-1} \mu_i^\alpha.$$

The cases  $\alpha = 0$  and  $\alpha = 1$  are trivial as  $\sigma_0 = n - 1$  and  $\sigma_1 = 2m$ , where  $m$  is the number of edges of  $G$ . Note that  $\sigma_{1/2}$  is equal to  $LEL$ . It is worth noting that  $n\sigma_{-1}$  is also equal to the Kirchhoff index of  $G$  (one can refer to the papers [2, 19, 37] for its definition and extensive applications in the theory of electric circuits, probabilistic theory and chemistry). Recently, various properties and the estimates of  $\sigma_\alpha$  have been well studied in the literature. For details, see [14, 31, 39, 41, 43].

Motivating the definitions of  $IE$ ,  $LEL$  and  $\sigma_\alpha$ , Akbari et al. [1] introduced the sum of the  $\alpha$ th powers of the signless Laplacian eigenvalues of  $G$  as

$$s_\alpha = s_\alpha(G) = \sum_{i=1}^n q_i^\alpha$$

and they also gave some relations between  $\sigma_\alpha$  and  $s_\alpha$ . In this sum, the cases  $\alpha = 0$  and  $\alpha = 1$  are trivial as  $s_0 = n$  and  $s_1 = 2m$ . Note that  $s_{1/2}$  is equal to the incidence energy  $IE$ . Note further that Laplacian eigenvalues and signless Laplacian eigenvalues of bipartite graphs coincide [7, 33, 34]. Therefore, for bipartite graphs  $\sigma_\alpha$  is equal to  $s_\alpha$  [3] and  $LEL$  is equal to  $IE$  [20]. Recently some properties and the lower and upper bounds of  $s_\alpha$  have been established in [1, 3, 27, 32].

In this paper, we obtain some new bounds on  $s_\alpha$  of bipartite graphs which improve the some bounds in [14]. In addition to this, we extend these bounds to non-bipartite graphs. As a result of these bounds, we also present some results on incidence energy.

## 2 Lemmas

Let  $t = t(G)$  denotes the number of spanning trees of  $G$ . Let  $\overline{G}$  be the complement of  $G$  and let  $G_1 \times G_2$  be the Cartesian product of the graphs  $G_1$  and  $G_2$  [6]. Now, we give two auxiliary quantities for a graph  $G$  as

$$t_1 = t_1(G) = \frac{2t(G \times K_2)}{t(G)} \text{ and } T = T(G) = \frac{1}{2} \left[ \Delta_1 + \delta + \sqrt{(\Delta_1 - \delta)^2 + 4\Delta_1} \right] \quad (1)$$

where  $\Delta_1$  and  $\delta$  are the maximum and the minimum vertex degrees of  $G$ , respectively.

**Lemma 2.1.** [25] *Let  $G$  be a graph with  $n$  vertices and  $m$  edges. Then*

$$\sum_{i=1}^n d_i^2 \leq m \left( \frac{2m}{n-1} + n - 2 \right). \quad (2)$$

Moreover, if  $G$  is connected, then the equality holds in (2) if and only if  $G$  is either a star  $K_{1,n-1}$  or a complete graph  $K_n$ .

**Lemma 2.2.** [7, 33, 34] *The spectra of  $L(G)$  and  $Q(G)$  coincide if and only if the graph  $G$  is bipartite.*

**Lemma 2.3.** [9] *If  $G$  is a connected bipartite graph of order  $n$ , then  $\prod_{i=1}^{n-1} q_i = \prod_{i=1}^{n-1} \mu_i = nt(G)$ . If  $G$  is a connected non-bipartite graph of order  $n$ , then  $\prod_{i=1}^n q_i = t_1(G)$ .*

**Lemma 2.4.** [5, 33] *Let  $G$  be a connected graph with  $n \geq 3$  vertices and maximum vertex degree  $\Delta_1$ . Then*

$$q_1 \geq T \geq \Delta_1 + 1$$

*with either equalities if and only if  $G$  is a star graph  $K_{1,n-1}$ .*

**Lemma 2.5.** [11] *Let  $G$  be a graph with second maximum vertex degree  $\Delta_2$ . Then*

$$q_2 \geq \Delta_2 - 1.$$

*If  $q_2 = \Delta_2 - 1$ , then the maximum and the second maximum vertex degrees are adjacent and  $\Delta_1 = \Delta_2$ .*

**Lemma 2.6.** [11] Let  $G$  be a connected graph with  $n$  vertices and minimum vertex degree  $\delta$ . Then

$$q_n < \delta.$$

**Lemma 2.7.** [8] Let  $G$  be a connected graph with diameter  $d(G)$ . If  $G$  has exactly  $k$  distinct signless Laplacian eigenvalues, then  $d(G) + 1 \leq k$ .

**Lemma 2.8.** [26] Let  $G$  be a connected graph with  $n \geq 3$  vertices and second maximum vertex degree  $\Delta_2$ . Then

$$\mu_2 \geq \Delta_2$$

with equality if  $G$  is a complete bipartite graph  $K_{p,q}$  or a tree with degree sequence  $\pi(T_n) = (n/2, n/2, 1, 1, \dots, 1)$ , where  $n \geq 4$  is even.

**Lemma 2.9.** [15] Let  $G$  be a graph with  $n$  vertices, different from  $K_n$  and let  $\delta$  be the minimum vertex degree of  $G$ . Then

$$\mu_{n-1} \leq \delta$$

**Lemma 2.10.** [12, 41] Let  $G$  be a simple graph with  $n$  vertices. Then  $\mu_1 = \mu_2 = \dots = \mu_{n-1}$  if and only if  $G \cong K_n$  or  $G \cong \overline{K}_n$ .

**Lemma 2.11.** [16] For  $a_1, a_2, \dots, a_n \geq 0$  and  $p_1, p_2, \dots, p_n \geq 0$  such that  $\sum_{i=1}^n p_i = 1$

$$\sum_{i=1}^n p_i a_i - \prod_{i=1}^n a_i^{p_i} \geq n\lambda \left( \frac{1}{n} \sum_{i=1}^n a_i - \prod_{i=1}^n a_i^{1/n} \right) \quad (3)$$

where  $\lambda = \min \{p_1, p_2, \dots, p_n\}$ . Moreover, equality holds in (3) if and only if  $a_1 = a_2 = \dots = a_n$ .

**Lemma 2.12.** [35] Let  $a_i > 0$ ,  $i = 1, 2, \dots, p$  be the  $p$  real numbers. Then

$$p(A_p - G_p) \geq (p-1)(A_{p-1} - G_{p-1}),$$

where

$$A_p = \frac{\sum_{i=1}^p a_i}{p} \text{ and } G_p = \left( \prod_{i=1}^p a_i \right)^{1/p}.$$

### 3 Main Results

In this section, we give the main results of the paper. First, we need the following lemma. For a graph  $G$  with signless Laplacian eigenvalues  $q_1 \geq q_2 \geq \dots \geq q_n$ , let

$$M_k = M_k(G) = \sum_{i=1}^k q_i$$

for  $1 \leq k \leq n-1$ . Then, we have:

**Lemma 3.1.** *Let  $G$  be a connected graph with  $n$  vertices and  $m$  edges.*

*i) If  $G$  is bipartite, then for  $1 \leq k \leq n - 2$*

$$M_k(G) \leq \frac{2mk + \sqrt{mk(n-k-1)(n^2 - n - 2m)}}{n-1} \quad (4)$$

*with equality holding in (4) if and only if  $G$  is either a star  $K_{1,n-1}$  or a complete graph  $K_n$  when  $k = 1$  and  $G$  is a complete graph  $K_n$  when  $2 \leq k \leq n - 2$ .*

*ii) If  $G$  is non-bipartite, then for  $1 \leq k \leq n - 1$*

$$M_k(G) \leq \frac{2mk + \sqrt{mk(n-k)(n^2 + \frac{2mn}{n-1} - 4m)}}{n} \quad (5)$$

*with equality holding in (5) if and only if  $G \cong K_n$  when  $k = 1$ .*

*Proof.* The inequality (4) was established in [40]. So we omit its proof here. Now we only prove the inequality (5). Let  $M_k = M_k(G)$ . It is clear that [7]

$$q_1 + q_2 + \cdots + q_n = 2m$$

and

$$q_1^2 + q_2^2 + \cdots + q_n^2 = 2m + \sum_{i=1}^n d_i^2$$

Then, using Cauchy-Schwarz inequality, we get

$$\begin{aligned} (2m - M_k)^2 &= (q_{k+1} + \cdots + q_n)^2 \\ &\leq (n - k)(q_{k+1}^2 + \cdots + q_n^2) \\ &= (n - k)\left(2m + \sum_{i=1}^n d_i^2 - (q_1^2 + \cdots + q_k^2)\right) \\ &\leq (n - k)\left(2m + \sum_{i=1}^n d_i^2 - \frac{1}{k}M_k^2\right). \end{aligned}$$

Therefore

$$M_k \leq \left\{ 2mk + \left[ k(n-k) \left( n \left( 2m + \sum_{i=1}^n d_i^2 \right) - 4m^2 \right) \right]^{1/2} \right\} / n. \quad (6)$$

From the inequality (6) and Lemma 2.1, the inequality (5) holds. Now we suppose that the equality holds in (5). Then, by Cauchy-Schwarz inequality we have  $q_1 = \cdots = q_k$  and  $q_{k+1} = \cdots = q_n$ . Since  $G$  is connected non-bipartite graph, by Lemma 2.1 and Lemma 2.7, we conclude that  $G \cong K_n$  when  $k = 1$ .  $\square$

The following result can be found in [14].

**Theorem 3.2.** [14] *Let  $G$  be a bipartite graph with  $n \geq 2$  vertices,  $m$  edges and positive integer  $k$  ( $1 \leq k \leq n - 2$ ).*

*(i) If  $0 < \alpha < 1$ , then*

$$s_\alpha(G) = \sigma_\alpha(G) \leq k^{1-\alpha} \left( \frac{2mk}{n-1} \right)^\alpha + (n-k-1)^{1-\alpha} \left( 2m - \frac{2mk}{n-1} \right)^\alpha \quad (7)$$

with equality holding in (7) if and only if  $G \cong K_n$  or  $G \cong \overline{K}_n$ .

(ii) If  $\alpha > 1$ , then

$$s_\alpha(G) = \sigma_\alpha(G) \geq k^{1-\alpha} \left( \frac{2mk}{n-1} \right)^\alpha + (n-k-1)^{1-\alpha} \left( 2m - \frac{2mk}{n-1} \right)^\alpha \quad (8)$$

with equality holding in (8) if and only if  $G \cong K_n$  or  $G \cong \overline{K}_n$ .

(iii) If  $G$  is connected and  $\alpha < 0$ , then

$$s_\alpha(G) = \sigma_\alpha(G) \leq \min_{1 \leq k \leq n-2} \left\{ k^{1-\alpha} \left[ \frac{2mk + \sqrt{mk(n-k-1)(n^2-n-2m)}}{n-1} \right]^\alpha + (n-k-1)^{1-\alpha} \left[ \frac{2mk(n-k-1) - \sqrt{mk(n-k-1)(n^2-n-2m)}}{n-1} \right]^\alpha \right\} \quad (9)$$

with equality holding in (9) if and only if  $G \cong K_{1,n-1}$  ( $k=1$ ) and  $G \cong K_n$  ( $2 \leq k \leq n-2$ ).

We now extend the above result to non-bipartite graphs.

**Theorem 3.3.** Let  $G$  be a non-bipartite graph with  $n \geq 2$  vertices,  $m$  edges and positive integer  $k$  ( $1 \leq k \leq n-1$ ).

(i) If  $0 < \alpha < 1$ , then

$$s_\alpha(G) \leq k^{1-\alpha} \left( \frac{2mk}{n} \right)^\alpha + (n-k)^{1-\alpha} \left( 2m - \frac{2mk}{n} \right)^\alpha \quad (10)$$

with equality holding in (10) if and only if  $G \cong \overline{K}_n$ .

(ii) If  $\alpha > 1$ , then

$$s_\alpha(G) \geq k^{1-\alpha} \left( \frac{2mk}{n} \right)^\alpha + (n-k)^{1-\alpha} \left( 2m - \frac{2mk}{n} \right)^\alpha \quad (11)$$

with equality holding in (11) if and only if  $G \cong \overline{K}_n$ .

(iii) If  $G$  is connected and  $\alpha < 0$ , then

$$s_\alpha(G) \leq \min_{1 \leq k \leq n-1} \left\{ k^{1-\alpha} \left[ \frac{2mk + \sqrt{mk(n-k)(n^2 + \frac{2mn}{n-1} - 4m)}}{n} \right]^\alpha + (n-k)^{1-\alpha} \left[ \frac{2m(n-k) - \sqrt{mk(n-k)(n^2 + \frac{2mn}{n-1} - 4m)}}{n} \right]^\alpha \right\} \quad (12)$$

with equality holding in (12) if and only if  $G \cong K_n$  when  $k=1$ .

*Proof.* Using power mean inequality, we get

$$\sum_{i=1}^k q_i^\alpha \leq k^{1-\alpha} \left( \sum_{i=1}^k q_i \right)^\alpha, \text{ as } 0 < \alpha < 1 \quad (13)$$

with equality holding in (13) if and only if  $q_1 = q_2 = \dots = q_k$ .

Considering the above manner, we also get

$$\sum_{i=k+1}^n q_i^\alpha \leq (n-k)^{1-\alpha} \left( 2m - \sum_{i=1}^k q_i \right)^\alpha, \text{ as } \sum_{i=1}^n q_i = 2m \text{ [7]}, \quad (14)$$

with equality holding in (14) if and only if  $q_{k+1} = q_{k+2} = \cdots = q_n$ . Since  $q_1 \geq q_2 \geq \cdots \geq q_n$ , we have

$$\frac{\sum_{i=1}^k q_i}{k} \geq \frac{\sum_{i=k+1}^n q_i}{n-k} = \frac{2m - \sum_{i=1}^k q_i}{n-k}.$$

Therefore, we get

$$\sum_{i=1}^k q_i \geq \frac{2mk}{n}. \quad (15)$$

By Eqs. (13) and (14), we obtain

$$\begin{aligned} s_\alpha(G) &= \sum_{i=1}^n q_i^\alpha = \sum_{i=1}^k q_i^\alpha + \sum_{i=k+1}^n q_i^\alpha \\ &\leq k^{1-\alpha} \left( \sum_{i=1}^k q_i \right)^\alpha + (n-k)^{1-\alpha} \left( 2m - \sum_{i=1}^k q_i \right)^\alpha. \end{aligned}$$

Now consider the following function

$$f(x) = k^{1-\alpha} x^\alpha + (n-k)^{1-\alpha} (2m-x)^\alpha$$

for  $x \geq \frac{2mk}{n}$ . Then it is easy to see that

$$f'(x) = \alpha \left[ \left( \frac{x}{k} \right)^{\alpha-1} - \left( \frac{2m-x}{n-k} \right)^{\alpha-1} \right] \leq 0, \text{ as } 0 < \alpha < 1.$$

Thus, by (15), we get

$$f(x) \leq f\left(\frac{2mk}{n}\right) = k^{1-\alpha} \left( \frac{2mk}{n} \right)^\alpha + (n-k)^{1-\alpha} \left( 2m - \frac{2mk}{n} \right)^\alpha.$$

Hence we get the the inequality (10). Now we suppose that the equality holds in (10). Then, from (13) and (14) we have  $q_1 = q_2 = \cdots = q_k$  and  $q_{k+1} = q_{k+2} = \cdots = q_n$ , respectively. Furthermore from (15), we have

$$\sum_{i=1}^k q_i = \frac{2mk}{n}.$$

Therefore

$$q_1 = q_2 = \cdots = q_n = \frac{2m}{n}.$$

Then, we conclude that  $G \cong \overline{K}_n$ .

Conversely, one can easily show that the equality holds in (10) for the complement of the complete graph  $\overline{K}_n$ .

(ii) Using power mean inequality, from (i), we obtain

$$s_\alpha(G) \geq k^{1-\alpha} \left( \sum_{i=1}^k q_i \right)^\alpha + (n-k)^{1-\alpha} \left( 2m - \sum_{i=1}^k q_i \right)^\alpha, \text{ as } \alpha > 1.$$

Note that  $f(x)$  is increasing function for  $x \geq \frac{2mk}{n}$  as  $\alpha > 1$ . Then, similar to the proof of (i), we get the inequality (11). Furthermore, the equality holds in (11) if and only if  $G \cong \overline{K}_n$ .

(iii) From Lemma 3.1, we have

$$\sum_{i=1}^k q_i \leq \frac{2mk + \sqrt{mk(n-k) \left( n^2 + \frac{2mn}{n-1} - 4m \right)}}{n}.$$

As  $\alpha < 0$ , from (i), we obtain that  $f(x)$  is increasing function for

$$\frac{2mk}{n} \leq x \leq \frac{1}{n} \left[ 2mk + \sqrt{mk(n-k) \left( n^2 + \frac{2mn}{n-1} - 4m \right)} \right].$$

Therefore

$$\begin{aligned} f(x) &\leq k^{1-\alpha} \left( \frac{2mk + \sqrt{mk(n-k) \left( n^2 + \frac{2mn}{n-1} - 4m \right)}}{n} \right)^\alpha + (n-k)^{1-\alpha} \\ &\quad \times \left( \frac{2m(n-k) - \sqrt{mk(n-k) \left( n^2 + \frac{2mn}{n-1} - 4m \right)}}{n} \right)^\alpha. \end{aligned}$$

Hence the inequality (12) holds. Now we suppose that the equality holds in (12). Therefore we get that

$$q_1 = q_2 = \cdots = q_k, q_{k+1} = q_{k+2} = \cdots = q_n$$

and

$$\sum_{i=1}^k q_i = \frac{2mk + \sqrt{mk(n-k) \left( n^2 + \frac{2mn}{n-1} - 4m \right)}}{n}.$$

Then, from Lemma 3.1, we conclude that  $G \cong K_n$  when  $k = 1$ .

Conversely, let  $G$  be isomorphic to the complete graph  $K_n$  when  $k = 1$ . Thus

$$\begin{aligned} &k^{1-\alpha} \left[ \frac{2mk + \sqrt{mk(n-k) \left( n^2 + \frac{2mn}{n-1} - 4m \right)}}{n} \right]^\alpha + (n-k)^{1-\alpha} \\ &\quad \times \left[ \frac{2m(n-k) - \sqrt{mk(n-k) \left( n^2 + \frac{2mn}{n-1} - 4m \right)}}{n} \right]^\alpha \\ &= (2(n-1))^\alpha + (n-1)(n-2)^\alpha, \text{ as } k = 1, m = n(n-1)/2 \\ &= s_\alpha(G), \text{ since } q_1 = 2(n-1), q_2 = \cdots = q_n = n-2. \end{aligned}$$



This completes the proof of theorem.  $\square$

**Theorem 3.4.** *Let  $\alpha$  be a real number with  $\alpha \neq 0, 1$  and let  $G$  be a connected graph with  $n \geq 3$  vertices and  $t$  spanning trees and also let  $t_1$  and  $T$  be given by (1). For any real number  $k \geq 0$ ,*

*i) if  $G$  is bipartite, then*

$$s_\alpha(G) = \sigma_\alpha(G) > (n-2)(nt)^{\alpha/(n-1)} \left[ \frac{(k+1)(nt)^{\alpha/[(k+1)(n-1)(n-2)]}}{T^{\alpha/[(k+1)(n-2)]}} - k \right] + T^\alpha. \quad (16)$$

*ii) If  $G$  is non-bipartite, then*

$$s_\alpha(G) > (n-1)(t_1)^{\alpha/n} \left[ \frac{(k+1)(t_1)^{\alpha/[(k+1)n(n-1)]}}{T^{\alpha/[(k+1)(n-1)]}} - k \right] + T^\alpha. \quad (17)$$

*Proof.* By Lemmas 2.2–2.4, 2.10 and 2.11, the inequality (16) can be proved using similar method of Theorem 3.4 in [14]. We now only prove the inequality (17).

Setting in Lemma 2.11  $a_i = q_i^\alpha$ ,  $i = 1, 2, \dots, n$  and

$$p_1 = \frac{k}{(k+1)n}, p_i = \frac{(k+1)n-k}{(k+1)n(n-1)}, i = 2, 3, \dots, n$$

we obtain

$$\begin{aligned} & \frac{kq_1^\alpha}{(k+1)n} + \frac{(k+1)n-k}{(k+1)n(n-1)} \sum_{i=2}^n q_i^\alpha - q_1^{\frac{k\alpha}{(k+1)n}} \prod_{i=2}^n q_i^{\frac{(k+1)n-k}{(k+1)n(n-1)}\alpha} \\ & \geq \frac{k}{(k+1)n} \sum_{i=1}^n q_i^\alpha - \frac{k}{k+1} \prod_{i=1}^n q_i^{\alpha/n}. \end{aligned}$$

Then, by Lemma 2.3, we have

$$\begin{aligned} & \frac{kq_1^\alpha}{(k+1)n} + \frac{(k+1)n-k}{(k+1)n(n-1)} (s_\alpha(G) - q_1^\alpha) - q_1^{-\frac{\alpha}{(k+1)(n-1)}} (t_1)^{\frac{(k+1)n-k}{(k+1)n(n-1)}\alpha} \\ & \geq \frac{k}{(k+1)n} s_\alpha(G) - \frac{k}{k+1} (t_1)^{\alpha/n}, \end{aligned}$$

i.e.,

$$s_\alpha(G) \geq (n-1) \left[ \frac{(k+1)(t_1)^{\frac{(k+1)n-k}{(k+1)n(n-1)}\alpha}}{q_1^{\frac{\alpha}{(k+1)(n-1)}}} + \frac{q_1^\alpha}{n-1} - k(t_1)^{\alpha/n} \right]. \quad (18)$$

Let us consider the auxiliary function

$$f(x) = \frac{(k+1)(t_1)^{\frac{(k+1)n-k}{(k+1)n(n-1)}\alpha}}{x^{\frac{\alpha}{(k+1)(n-1)}}} + \frac{x^\alpha}{n-1}.$$

It is easy to see that  $f(x)$  is increasing for  $x > (t_1)^{1/n}$  whether  $\alpha > 0$  or  $\alpha < 0$ . By Lemmas 2.3, 2.4 and Theorem 3.3 in [4], we have

$$q_1 \geq T \geq \Delta_1 + 1 > \Delta_1 \geq \frac{2m}{n} \geq (t_1)^{1/n}$$

Therefore

$$f(x) \geq f(T) = \frac{(k+1)(t_1)^{\frac{(k+1)n-k}{(k+1)n(n-1)}\alpha}}{T^{\frac{\alpha}{(k+1)(n-1)}}} + \frac{T^\alpha}{n-1}.$$

Combining this with (18) we get the inequality (17). Now we assume that the equality holds in (17). Then all inequalities in the above arguments must be equalities. Thus  $q_1 = T$  and  $q_1 = q_2 = \dots = q_n = \frac{2m}{n}$ . Thus we have that  $q_1 = \frac{2m}{n} \leq \Delta_1 < \Delta_1 + 1 \leq T$  which contradicts with the result in Lemma 2.4 [4]. Hence (17) cannot become an equality.  $\square$

**Remark 3.5.** By Lemmas 2.2 and 2.4, we have that  $\mu_1 = q_1 \geq T \geq \Delta_1 + 1$  for bipartite graphs. Then from the proof of Theorem 3.4 in [14], one can arrive at the bound (16) improves the bound of Theorem 3.4 in [14] for bipartite graphs.

Taking  $k = 1$  in Theorem 3.4, we have the following result.

**Corollary 3.6.** Let  $\alpha$  be a real number with  $\alpha \neq 0, 1$  and let  $G$  be a connected graph with  $n \geq 3$  vertices and  $t$  spanning trees and also let  $t_1$  and  $T$  be given by (1).

i) if  $G$  is bipartite, then

$$s_\alpha(G) = \sigma_\alpha(G) > (n-2)(nt)^{\alpha/(n-1)} \left[ \frac{2(nt)^{\alpha/[2(n-1)(n-2)]}}{T^{\alpha/[2(n-2)]}} - 1 \right] + T^\alpha. \quad (19)$$

ii) If  $G$  is non-bipartite, then

$$s_\alpha(G) > (n-1)(t_1)^{\alpha/n} \left[ \frac{2(t_1)^{\alpha/[2n(n-1)]}}{T^{\alpha/[2(n-1)]}} - 1 \right] + T^\alpha. \quad (20)$$

As in Remark 3.5, one can easily conclude that the bound (19) of Corollary 3.6 improves Corollary 3.5 in [14]. Moreover, taking  $\alpha = 1/2$  in Corollary 3.6, we have the following result.

**Corollary 3.7.** [4] Let  $G$  be a connected graph with  $n \geq 3$  vertices and  $t$  spanning trees and also let  $t_1$  and  $T$  be given by (1).

i) if  $G$  is bipartite, then

$$IE(G) = LEL(G) > \sqrt{T} + (n-2)(nt)^{1/[2(n-1)]} \left[ \frac{2(nt)^{1/[4(n-1)(n-2)]}}{T^{1/[4(n-2)]}} - 1 \right]. \quad (21)$$

ii) if  $G$  is non-bipartite, then

$$IE(G) > \sqrt{T} + (n-1)(t_1)^{1/(2n)} \left[ \frac{2(t_1)^{1/[4n(n-1)]}}{T^{1/[4(n-1)]}} - 1 \right]. \quad (22)$$

**Theorem 3.8.** Let  $\alpha$  be a real number with  $\alpha \neq 0, 1$  and let  $G$  be a connected graph with  $n \geq 3$  vertices and  $t$  spanning trees and also  $t_1$  and  $T$  be given by (1).

i) if  $G$  is bipartite, then

$$s_\alpha(G) = \sigma_\alpha(G) \geq T^\alpha + (n-2) \left( \frac{nt}{T} \right)^{\alpha/(n-2)} + \left( \Delta_2^{\alpha/2} - \delta^{\alpha/2} \right)^2. \quad (23)$$

ii) if  $G$  is non-bipartite, then

$$s_\alpha(G) > T^\alpha + (n-1) \left( \frac{t_1}{T} \right)^{\alpha/(n-1)} + \left( (\Delta_2 - 1)^{\alpha/2} - \delta^{\alpha/2} \right)^2 \quad (24)$$

where  $\Delta_2$  and  $\delta$  are the second maximum and the minimum vertex degrees of the graph  $G$ , respectively.

*Proof.* Using Lemmas 2.2–2.4, 2.8, 2.9 and 2.12, one can prove inequality (23) similar to the proof of Theorem 3.9 in [14]. Here we only prove the inequality (24).

By Lemma 2.12, we have

$$p(A_p - G_p) \geq (p-1)(A_{p-1} - G_{p-1}) \geq \cdots \geq 2(A_2 - G_2)$$

i.e.,

$$A_p \geq G_p + \frac{2}{p} \left( \frac{a_1 + a_2}{2} - \sqrt{a_1 a_2} \right) = G_p + \frac{1}{p} (\sqrt{a_1} - \sqrt{a_2})^2 \quad (25)$$

see, [14]. Setting  $p = n-1$ ,  $(a_1, a_2, \dots, a_{n-1}) = (q_2^\alpha, q_3^\alpha, \dots, q_n^\alpha)$  and  $a_1 = q_2^\alpha$ ,  $a_2 = q_n^\alpha$  in (25), we obtain

$$s_\alpha(G) = \sum_{i=1}^n q_i^\alpha \geq q_1^\alpha + (n-1) \left( \prod_{i=2}^n q_i \right)^{\alpha/(n-1)} + \left( q_2^{\alpha/2} - q_n^{\alpha/2} \right)^2.$$

Considering Lemmas 2.3, 2.5 and 2.6, we have

$$s_\alpha(G) = \sum_{i=1}^n q_i^\alpha \geq q_1^\alpha + (n-1) \left( \frac{t_1}{q_1} \right)^{\alpha/(n-1)} + \left( (\Delta_2 - 1)^{\alpha/2} - \delta^{\alpha/2} \right)^2. \quad (26)$$

Let us consider the auxiliary function

$$f(x) = x^\alpha + (n-1) \left( \frac{t_1}{x} \right)^{\alpha/(n-1)}.$$

Note that  $f(x)$  is increasing for  $x > (t_1)^{1/n}$  for both  $\alpha > 0$  and  $\alpha < 0$  [3]. Then by Lemmas 2.3, 2.4 and Theorem 4.9 in [3], we have

$$f(x) \geq f(T) = T^\alpha + (n-1) \left( \frac{t_1}{T} \right)^{\alpha/(n-1)}.$$

Combining this with Eq. (26), we get the inequality (24).

**Remark 3.9.** By Lemmas 2.2 and 2.4, we have that  $\mu_1 = q_1 \geq T \geq \Delta_1 + 1$  for bipartite graphs. Then, from the proof of Theorem 3.9 in [14], one can arrive at the bound (23) improves the bound of Theorem 3.9 in [14] for bipartite graphs. Moreover, it is clear that the results of Theorem 3.8 are better than the results of Theorem 4.9 in [3].

□

Taking  $\alpha = 1/2$  in Theorem 3.8, we get the following result on  $IE$ .

**Corollary 3.10.** *Let  $G$  be a connected graph with  $n \geq 3$  vertices and  $t$  spanning trees and also let  $t_1$  and  $T$  be given by (1).*

*i) if  $G$  is bipartite, then*

$$IE(G) = LEL(G) \geq \sqrt{T} + (n-2) \left( \frac{nt}{T} \right)^{1/(2(n-2))} + \left( \Delta_2^{1/4} - \delta^{1/4} \right)^2. \quad (27)$$

*ii) if  $G$  is non-bipartite, then*

$$IE(G) > \sqrt{T} + (n-1) \left( \frac{t_1}{T} \right)^{1/(2(n-1))} + \left( (\Delta_2 - 1)^{1/4} - \delta^{1/4} \right)^2. \quad (28)$$

where  $\Delta_2$  and  $\delta$  are the second maximum and the minimum vertex degrees of the graph  $G$ , respectively.

**Remark 3.11.** *It is clear that the results of Corollary 3.10 improve the results of Theorem 4.8 in [3].*

**Remark 3.12.** *We finally note that, if we can establish a new lower bound such that  $q_1 \geq \beta \geq T$ , then we can improve the results in Theorems 3.4 and 3.8.*

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